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Positons for the Toda lattice and related spectral problems

A A Stahlhofen and V B Matveev†

Max-Planck-Institut Metallforschung, Heisenbergstrasse 1, D-70569 Stuttgart, Germany

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Abstract. The concept of positons is introduced for the Toda lattice equation. It is shown that these multiparametric oscillating and slowly decaying solutions, when inserted as potentials in the finite-difference Schrödinger equation of the corresponding Lax pair, lead to a trivial S-matrix. The resulting eigenvalues are embedded in the continuous spectrum of this infinite Jacobi matrix. The singularities connected with the one-positon solution are discussed and compared with those of the positons of continuous integrable models. The special features of the soliton–positon interaction are analysed.

1. Darboux transformations of the Toda lattice

The Toda lattice equation reads

$$\ddot{x}_n = \exp(x_{n-1} - x_n) - \exp(x_n - x_{n+1}) \tag{1}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Introduction of velocities $v_n := \dot{x}_n$ and new dependent variables $u_n := \exp(x_n - x_{n+1})$ allows (1) to be represented as the system

$$\dot{v}_n = u_{n-1} - u_n \tag{2}$$

$$\dot{u}_n = u_n(v_n - v_{n+1}). \tag{3}$$

This system has the Lax representation

$$\dot{L} = [A, L] \tag{4}$$

where the elements are given by

$$L_{nm} = v_n\delta_{nm} + u_n\delta_{n+1,m} + \delta_{n-1,m} \tag{5}$$

$$A_{nm} = v_n\delta_{nm} + \delta_{n+1,m}. \tag{6}$$

The discussion below focuses on the eigenvalue problem resulting from (5). It should be noted, however, that it is possible to reduce L to a Hermitian Jacobi matrix H via a gauge transformation using for this purpose a diagonal matrix G with $G_{nm} = e^{x_n/2}\delta_{nm}$. This leads to

$$H = G^{-1}LG \tag{7}$$

$$H_{nm} = v_n\delta_{nm} + w_n\delta_{n+1,m} + w_{n-1}\delta_{n-1,m} \tag{8}$$

with $w_n = \exp(x_n/2)/\exp(x_{n+1}/2)$. The transformation of the corresponding eigenvalue problems is discussed below (cf equation (32)). The matrix operator H constructed in (7),

† Permanent address: St Petersburg branch of the Steklov Mathematical Institute, Fontanka 27, St Petersburg 191011, Russia.

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(8) is similar to the corresponding operator(s) introduced in [1, 2], where the Toda lattice (1) has been studied via the inverse scattering method.

The Lax equation (4) with the definitions (5), (6) represents a consistency condition for the coupled linear system

$$\partial_t f_n = v_n f_n + f_{n-1} \tag{9}$$

$$\lambda f_n = v_n f_n + u_n f_{n+1} + f_{n-1}. \tag{10}$$

The formalism of Darboux transformations (DTs) for the Toda lattice (1) to be used in the following has been introduced in [3], elucidated and applied to (1) in [4] and summarized in [5]. The strategy is as follows: let $\phi_n(1), \dots, \phi_n(N)$ be fixed linear independent solutions of (9), (10) corresponding to different values $\lambda_1, \dots, \lambda_N$ of λ. Then the following theorem holds (cf [4]): the system (9), (10) is covariant with respect to the 'difference DT $f_n \to \psi_n[N]$' with

$$\psi_n[N] = \frac{\mathrm{Cas}[\phi_n(1), \dots, \phi_n(N), f_n]}{\mathrm{Cas}[\phi_{n+1}(1), \dots, \phi_{n+1}(N)]} \tag{11}$$

the Casorati determinant Cas[.] is defined as $(j, k = 1, \dots, N)$

$$\mathrm{Cas}[g_n(1), \dots, g_n(N)] := \det A \qquad A_{jk} = g_{n+j-1}(k) \tag{12}$$

and f_n denotes the general solution of (9), (10) with eigenvalue λ. Covariance means that $\psi[N]$ solves a system with the same structure as (9), (10) with new coefficients $u_n[N]$, $v_n[N]$:

$$\partial_t \psi_n[N] = v_n[N]\psi_n[N] + \psi_{n-1}[N] \tag{13}$$

$$\lambda\psi_n[N] = v_n[N]\psi_n[N] + u_n[N]\psi_{n+1}[N] + \psi_{n-1}[N]. \tag{14}$$

The new coefficients $u_n[N]$, $v_n[N]$ and the new solution $x_n[N]$ are easily computed in terms of the old coefficients u_n, v_n and the functions $\phi_n(1), \dots, \phi_n(N)$:

$$v_n[N] = v_{n+N} + \partial_t \log \frac{\mathrm{Cas}[\phi_n(1), \dots, \phi_n(N)]}{\mathrm{Cas}[\phi_{n+1}(1), \dots, \phi_{n+1}(N)]} \tag{15}$$

$$u_n[N] = u_{n+N} \frac{\mathrm{Cas}[\phi_n(1), \dots, \phi_n(N)]\mathrm{Cas}[\phi_{n+2}(1), \dots, \phi_{n+2}(N)]}{\mathrm{Cas}^2[\phi_{n+1}(1), \dots, \phi_{n+1}(N)]} \tag{16}$$

$$x_n[N] = x_{n+N} + \frac{1}{2} \log \frac{\mathrm{Cas}^2[\phi_n(1), \dots, \phi_n(N)]}{\mathrm{Cas}^2[\phi_{n+1}(1), \dots, \phi_{n+1}(N)]}. \tag{17}$$

Here we consider only the simplest case of trivial background setting

$$x_n = 0 \qquad u_n = 1 \qquad v_n = 0. \tag{18}$$

In this case equations (9), (10) simplify to

$$\partial_t f_n = f_{n-1} \tag{19}$$

$$f_{n+1} + f_{n-1} = \lambda f_n. \tag{20}$$

The general solution of (19), (20) is

$$f_n = c_1 \omega^n e^{t/\omega} + c_2 \omega^{-n} e^{t\omega} \tag{21}$$

this leads via

$$\omega^2 - \lambda\omega + 1 = 0 \tag{22}$$

to the eigenvalue

$$\lambda = \omega + \omega^{-1}. \tag{23}$$

2. The soliton solution of the Toda lattice

A well known construction of the n-soliton solutions of (1) can be found, for example, in [6]. We do not follow this derivation here, but derive the soliton solution of (1) and discuss its spectral properties in the framework of DTs in order to illustrate this formalism as sketched above. Starting for definiteness from a trivial background, the soliton of the Toda lattice (1) is obtained from setting $N = 1$ in (12)–(14) and constructing solutions of (19), (20) with $|\lambda| > 2$. These are obtained by setting $\omega = \exp(k_1)$ (or $\omega = \exp(-k_1)$) in (21)—which leads to the eigenvalue $\lambda = \pm 2\cosh k_1$—and read

$$\begin{aligned}\phi_n^{\text{sol}}(1) &= \exp(t\cosh k_1)\cosh(k_1 n \mp t\sinh k_1 + x_1)\\ &:= \exp(t\cosh k_1)\cosh(Y_n);\ x_1, k_1 \in \mathbb{R}.\end{aligned} \tag{24}$$

Thus the explicit form of the soliton solution follows from

$$x_n^{\text{sol}} = \tfrac{1}{2}\log\left(\frac{\phi_n^{\text{sol}}(1)}{\phi_{n+1}^{\text{sol}}(1)}\right)^2 \tag{25}$$

the sign ambiguity induced by (24) corresponds to the two possible directions of propagation.

The soliton solution described by (25) tends asymptotically to constant values, i.e.

$$x_n^{\text{sol}} = \mp k_1 \qquad n \to \pm\infty. \tag{26}$$

The reflection—and transmission—coefficients associated with the soliton (25) can be determined as usual by the Jost solutions f_n of (20) which are fixed by the asymptotic condition

$$f_n = \mathrm{e}^{\mathrm{i}kn}[1 + o(1)] \qquad n \to +\infty \tag{27}$$

the accompanying definition

$$f_n = a(k)\mathrm{e}^{\mathrm{i}kn} + b(k)\mathrm{e}^{-\mathrm{i}kn} \qquad n \to -\infty \tag{28}$$

introduces the reflection coefficient $b(k)$ and the transmission coefficient $a(k)$.

These coefficients are obtained by comparing the asymptotic form $\psi_{n\to\infty}$ of the ψ-function,

$$\psi_n := \frac{\mathrm{Cas}\left[\phi_n^{\text{sol}}(1), \mathrm{e}^{\mathrm{i}kn}\right]}{\phi_{n+1}^{\text{sol}}(1)} \tag{29}$$

with the definitions. Since these calculation are based on arbitrary but *fixed* time, the time-dependent terms in $\exp(\mathrm{i}kn)$ in (29) can be dropped for simplicity as has been done here. Equation (29) leads according (24) to

$$\begin{aligned}\psi_n &= \mathrm{e}^{\mathrm{i}kn}\left(\mathrm{e}^{\mathrm{i}k}\frac{\cosh Y_n}{\cosh Y_{n+1}} - 1\right)\\ &= \mathrm{e}^{\mathrm{i}kn}(\mathrm{e}^{\mathrm{i}k\mp k_1} - 1) \qquad n \to \pm\infty.\end{aligned} \tag{30}$$

Hence we can conclude that the Jost function in the soliton case is given by

$$f_n = (\mathrm{e}^{\mathrm{i}k-k_1} - 1)^{-1}\psi_n. \tag{31}$$

The solution of

$$H\psi = \lambda\psi \tag{32}$$

where H has been defined in (8), follows from $Lf = \lambda f$ via $\psi = G^{-1}f$, i.e. $\psi_n = \exp(-x_n/2)f_n$. The corresponding Jost solution has to be multiplied accordingly. In the soliton case discussed here, this factor tends to (cf (26)) $\exp(\pm k_1/2)$ for $n \to \pm\infty$.

A comparison of (30), (31) with (28) shows that the soliton (25) is for fixed time a reflectionless potential when inserted in (8); the corresponding transmission coefficient is given by

$$a_L(k) = -\exp(k_1 + ik)\frac{\exp(-k_1 - ik) - 1}{\exp(-k_1 + ik) - 1}. \tag{33}$$

It can be seen from (33) that the transmission coefficient of L, $a_L(k)$, contains a non-unitary factor e^{k_1}; this factor vanishes when the derivation is based on the self-adjoint operator H from (8). An extension of this construction to n solitons is easily achieved following the strategy of the DT for the Toda lattice summarized in [5].

3. Difference operators and trivial monodromy

Difference operators of the form (7), (8), obtained by Darboux dressing of the zero background, give rise to the following two new phenomena.

(i) It is possible to construct explicitly solvable non-singular purely periodic difference operators having an arbitrary period, i.e. $u_{n+m} = u_n$, $v_{n+m} = v_n$ for any integer m, and trivial monodromy. In the difference case m periodic potentials (α) correspond to a spectrum having m finite gaps and (β) are expressed in terms of the m-dimensional Riemann theta functions of hyperelliptic curves of genus m. Thus the examples discussed in the following correspond to a degenerate case of these finite gap potentials obtained by a limit in which the length of the gaps tends to zero.

(ii) A one-step Darboux dressing of zero background with an oscillating starting solution leads to solvable almost periodic potentials which are, in principle, unbounded on the whole axis. The difference from case (i) is fixed by the choice of the spectral parameter k_1 in the generating solution φ, where $\varphi = \sin(k_1 n + x_1)$ corresponds to the zero background. The associated Floquet–Bloch solutions read (cf the corresponding ψ-function for the soliton case considered in the previous section)

$$\psi_{1,2} = e^{\pm ikn}\left(e^{ik}\frac{\sin(k_1 n + x_1)}{\sin(k_1(n+1) + x_1)} - 1\right). \tag{34}$$

It should be noted again that the discussion is always up to a factor $\exp(-x_n/2)$ resulting from the transformation from L (equation (5)) to H (equations (7), (8)). If $k_1 = \pi r$ with r a rational number, then $\psi_{1,2}$ in (34) is obviously non-singular and periodic. If $k_1 = \pi/m$, for instance, the period of the coefficients of (8) will be exactly $2m$. In this case it is clear that a generic choice of the phase x_1, $x_1 \neq 0$, leads to globally bounded Floquet–Bloch solutions corresponding to the trivial dependence of the quasi-momentum on energy. If, by contrast, the relation $k_1 = \pi q$ with q irrational holds, then it is always possible to choose n and m such that

$$|k_1 n + x_1 - \pi m| < \epsilon. \tag{35}$$

This means that the potential and the ψ-function obtained in the construction are unbounded while having no singularities. In this case it is obvious that the resulting operators are not periodic but almost periodic. These results follow from the Dirichlet theorem about the approximation of irrational numbers by rational numbers† proven, for example, in [7]. In this context it is interesting to note that the explicit examples constructed below are

† One of the authors (VBM) is indepted to Professor M Knörrer for his arguments provided in the context of discussions about the spacetime discretized sinh–Gordon equation.

apparently not discussed in the general theory of almost periodic operators as the recent review [8] indicates.

In the continuous case, by comparison, one has the following situation: it has been shown in [9], that in one dimension no smooth Schrödinger operator $-\partial_{xx} + u(x)$ having non-trivial potential $u(x)$ and no gaps in the spectrum exists. If the condition of non-singularity is dropped, however, the construction of the Hill operator with singular potential and no gaps is well established. The associated potential

$$u(x) = \frac{2\kappa^2}{\sin(x + x_1)^2} \tag{36}$$

has second-order poles on the real axis. This potential can be considered to result from a degeneration of the periodic potential

$$u(x) = 2\mathcal{P}(x) + c \tag{37}$$

with one gap which corresponds to the case when the pure imaginary period tends to infinity or to a closure of the gap.

The singular potential (36) can be deduced even more easily by one DT of the Schrödinger equation for trivial background via

$$u(x) = -2\partial_{xx}\log(\varphi(x, k_1)) := -2\partial_{xx}\log(\sin(k_1(x + x_1))). \tag{38}$$

The associated Floquet–Bloch solutions are then given by

$$\psi_{1,2} = \mathrm{e}^{\pm \mathrm{i}kx}(\pm\mathrm{i}k - k_1\cot(k_1(x + x_0))). \tag{39}$$

Instead of the usual structure

$$\psi_{1,2} = \mathrm{e}^{\pm\mathrm{i}p(k)x}\chi_{1,2}(x, k),\ \chi_{1,2}(x + T) = \chi_{1,2}(x),\ T := \frac{\pi}{k_1} \tag{40}$$

characteristic for the generic case with a nonlinear dependence of p on k we have for this special case a linear dependence of p on k. Thus the monodromy matrix in the basis of Floquet–Bloch solutions is given by

$$M = \begin{pmatrix} \exp(+\mathrm{i}k\pi/k_1 & 0 \\ 0 & \exp(-\mathrm{i}k\pi/k_1) \end{pmatrix}. \tag{41}$$

4. Positons of the Toda lattice

The positon solutions (briefly: positons) of the Toda lattice (1) are generated using bounded solutions of (20) corresponding to the continuous spectrum of the free operators L_0 (or H_0), i.e. to $-2 \leqslant \lambda \leqslant 2$. Restricting ourselves to this case we set $\omega = \exp(\mathrm{i}k)$, $\lambda = 2\cos(k)$ in (23). (Note in analogy to the soliton case, equation (24), the choice $\omega = \exp(-\mathrm{i}k)$ is also possible.) The solutions of (19), (20) thus read for a trivial background

$$\begin{aligned}\phi_n(j) &= \exp(t\cos k_j)\sin(k_j n \mp t\sin k_j + p_j(k_j)) \\ &= \exp(t\cos k_j)\sin(T_n^{(j)})\end{aligned} \tag{42}$$

with real k_j, p_j. To be specific, the positons result from (i) a two-step DT (i.e. $N = 2$ in (15)–(17)) with two functions $\phi_n(j)$, $(j = 1, 2)$ of type (42) and (ii) subsequently computing the limit $k_2 \to k_1$ in the result of this DT. This ansatz leads to oscillating long-ranged solutions where the conventional scheme of inverse scattering based on exponential decay (cf [1, 2]) is not applicable.

Following the strategy outlined above we first write (17) as

$$x_n[2] = \tfrac{1}{2}\log\frac{\mathrm{Cas}^2[\phi_n(1),\phi_n(2)]}{\mathrm{Cas}^2[\phi_{n+1}(1),\phi_{n+1}(2)]} := \frac{1}{2}\log\left(\frac{\tau_n}{\tau_{n+1}}\right)^2. \tag{43}$$

With (42) one obtains from (43)

$$\tau_n = \exp[t(\cos k_1 + \cos k_2)](\sin(T_n^{(1)})\sin(T_n^{(2)}+k_2) - \sin(T_n^{(2)})\sin(T_n^{(1)}+k_1)). \tag{44}$$

We now assume that $\phi_n(2)$ and $p_2(k_2)$ are analytical functions of the spectral parameter. The limit $k_2 \to k_1$ gives for the corresponding Taylor expansion of $\phi_n(2)$ the expression

$$\phi_n(2) = \phi_n(1) + \left.\frac{\partial\phi_n(2)}{\partial k_2}\right|_{k_2=k_1} \triangle k + \cdots \tag{45}$$

which leads upon insertion of (42) to

$$\phi_n(2) = \exp(t\cos k_1)\sin(T_n^{(1)}) + [-t\sin(k_1)\sin(T_n^{(1)}) + g_n\cos(T_n^{(1)})] \times \exp(t\cos k_1)\triangle k + \cdots \tag{46}$$

where higher-order terms have been omitted and the abbreviations

$$\triangle k = k_2 - k_1,\; g_n = \partial_{k_1} T_n^{(1)} = n \mp t\cos k_1 + \tilde{p}_1 \tag{47}$$

with $\tilde{p}_1 = \partial_{k_1} p_1$ have been introduced.

When inserting the expansion (46) in the ansatz (43), one can verify in a straightforward calculation that the n-independent factors $\exp(t\cos k_1)$ and $\triangle k$ cancel in the ratio. This allows us to define the one-positon solution of the Toda lattice (1) as

$$x_n := \tfrac{1}{2}\log\frac{\mathrm{Cas}^2[\phi_n(1),\partial_{k_1}\phi_n(1)]}{\mathrm{Cas}^2[\phi_{n+1}(1),\partial_{k_1}\phi_{n+1}(1)]} = \tfrac{1}{2}\log\left(\frac{\tau_n^{\mathrm{pos}}}{\tau_{n+1}^{\mathrm{pos}}}\right)^2 \tag{48}$$

the positon τ-function τ_n^{pos} reads explicitly as

$$\tau_n^{\mathrm{pos}} = -(g_n + \tfrac{1}{2})\sin k_1 + \tfrac{1}{2}\sin(2T_n + k_1) \tag{49}$$

with the simplified notation

$$\phi_n(1) = \sin(T_n) \qquad T_n = k_1 n \mp t\sin k_1 + p_1(k_1) \qquad g_n = \partial_{k_1} T_n = n \mp t\cos k_1 + \tilde{p}_1. \tag{50}$$

While (49) is completely sufficient to analyse the positon (48), it should be stressed here that the form of the τ-function τ_n^{pos} is not unique since the positon (48) can be defined only up to constant terms. It is also possible to take the τ-function resulting from the soliton–positon solution of (1) in the limit of the vanishing spectral parameter of the soliton as the τ-function of the positon. The explicit form of this alternative definition is derived below.

Any singularities of the positon (48) are determined by the zeros of τ_n^{pos} in (49). A short inspection shows, that for any fixed t, the right-hand side of (49) can only vanish for a few values of n. This can be verified by choosing simple values for k_1 and all phases and to analysing the resulting expressions. (The choice $k_1 = \frac{1}{2}\pi$, for instance, makes g_n in (49) time-independent and is easy to analyse.) The general observation is that it is always possible to choose the phases p_1 and $\tilde{p}_1$ such that (49) will be singular only at some values of t for every lattice site n.

At this point it is worthwhile to summarize the singularity structure of positons in the cases studied so far: the positon of the KdV equation is singular (cf [10, 11]). It has a second-order pole propagating from the right to the left. The velocity of the pole (i) oscillates periodically around a constant average value fixed by the spectral parameter k of the positon and (ii) becomes infinite at certain periodically occurring points of the (x, t) plane (cf [13]). The positon of the modified KdV equation [12] has two first-order poles with a mutual distance that oscillates periodically; the moments of infinite velocity occur in this case also. The sinh–Gordon positon [14] is also singular except at a discrete sequence of equidistantly spaced values of time. The non-local KdV equation provides a first exception: the complex valued positon remains non-singular almost all the time [15]. The space and time discretized sinh–Gordon equation [16] has positon solutions which are non-singular for certain values of energy. Thus the positon (48) of the Toda lattice (1) could be coined 'weakly singular' when compared with the positons of continuous integrable systems.

The ψ-function of the Lax pair (9), (10) associated with the positon (48) is obtained by computing the same limit in the spectral parameter in the ansatz (11). Omitting again for simplicity the arbitrary but fixed time dependence, the expression

$$\psi_n = \frac{\mathrm{Cas}[\sin T_n, g_n \cos T_n, \mathrm{e}^{ikn}]}{\mathrm{Cas}[\sin(T_n + k_1), (g_n + 1)\cos(T_n + k_1)]} \tag{51}$$

has to be evaluated. The leading term of asymptotics of ψ_n for $n \to \pm\infty$ reads

$$\psi_n = (\mathrm{e}^{2ik} - 2\mathrm{e}^{ik}\cos k_1 + 1)\mathrm{e}^{ikn}[1 + \mathrm{O}(1/n)]. \tag{52}$$

An inspection of (51), (52) shows that for $k = \pm k_1$ the ψ-function ψ_n is square summable, i.e. the series $\sum_{-\infty}^{+\infty} \mid \psi_n^2 \mid$ is convergent; ψ_n oscillates for all other real values of k. Thus $\lambda = 2\cos k_1$ is a discrete eigenvalue of the operator L (or H) embedded in the continuous spectrum. This conclusion is supported by the asymptotics of the positon (48) reading

$$x_n^{\mathrm{pos}} = \tfrac{1}{2}\log\left(1 - \frac{1}{2\sin k_1}\frac{\sin(2(T_n + k_1))}{n}\right)^2 \qquad n \to \pm\infty \tag{53}$$

the oscillating and slowly decaying term appearing on the right-hand side of (53) is characteristic for a von Neumann–Wigner potential supporting embedded eigenvalues. The asymptotic definition (27) of the Jost function f_n and the asymptotic form (52) of the ψ-function ψ_n show that the Jost solution associated with the positon is given by

$$f_n = (\mathrm{e}^{2ik} - 2\mathrm{e}^{ik}\cos k_1 + 1)^{-1}\psi_n. \tag{54}$$

The Jost solution of the corresponding equation $H\psi = \lambda\psi$ results from (53) via multiplication by the factor $\exp(-x_n/2)$, which tends to a constant for $n \to \pm\infty$ in (48). When (52), (53) are compared with the definition (28) of the reflection and transmission coefficients, one sees immediately that the positon (48) leads to the identities $a(k) = 1$, $b(k) = 0$. In other words: the scattering matrix associated with the operator L (or H) is the unit matrix for the potential defined for fixed t by the positon solution (48) of the Toda lattice (1). This is a special feature common to all potentials resulting from positon solutions of nonlinear integrable systems which we suggest calling 'supertransparent' (cf [10]–[14]). Since, however, the supertransparency phenomenon is, in the present lattice case, related to genuine eigenvalues in the continuum instead of singular positive eigenvalues in the continuum (coined 'speics' in [10]), the supertransparency looks very interesting with respect to possible applications.

We now discuss the solution of (1) describing the soliton–positon interaction. It reads in compact form:

$$x_n = \tfrac{1}{2}\log\left(\frac{\tau_n^{\mathrm{sp}}}{\tau_{n+1}^{\mathrm{sp}}}\right)^2 \tag{55}$$

with

$$\tau_n^{\mathrm{sp}} = \mathrm{Cas}[\sin(T_n), g_n\cos(T_n), \cosh(Y_n)] \tag{56}$$

$$T_n = k_1 n \mp t\sin k_1 + p_1 \qquad g_n = n \mp t\cos k_1 + \tilde{p}_1 \tag{57}$$

$$Y_n = \gamma n \mp t\sinh\gamma + p_2 \qquad k_1, \gamma, n, p_i, \tilde{p}_1 \in \mathbb{R}. \tag{58}$$

Note, that the time-independent factors cancelling in the calculation of x_n have again been dropped. In order to determine the results of the interaction we have subsequently to keep Y_n and g_n fixed while computing the limit $t \to \pm\infty$; the results give the asymptotic forms of the soliton (Y_n fixed) and the positon (g_n fixed).

In the soliton case, the leading term of asymptotics of the τ-function τ_n^{sp} from (56) has the form of the soliton-generating function (24). This means that the soliton is not changed by the collision. The positon, however, acquires two 'phase shifts' and a multiplicative factor which are functions of the spectral parameters of the soliton and positon. In the area where the positon is concentrated, the τ-function τ_n^{sp} becomes

$$\tau_n^{\mathrm{sp}} = \tfrac{1}{2}\mathrm{e}^{Y_n}\,\Delta_1\,\{\sin k_1[-(g_n + \tfrac{1}{2}) + \Delta_2] + \tfrac{1}{2}\sin(2T_n + k_1 + \Delta_3)\} \tag{59}$$

the multiplicative factor Δ_1 and the phases Δ_2, Δ_3 are given by

$$\Delta_1 = (1 - 2\mathrm{e}^{\gamma}\cos k_1 + \mathrm{e}^{2\gamma}) \tag{60}$$

$$\Delta_2 = \frac{\mathrm{e}^{\gamma}\cos k_1 - 1}{1 - 2\mathrm{e}^{\gamma}\cos k_1 + \mathrm{e}^{2\gamma}} \tag{61}$$

$$\Delta_3 = \arctan\frac{\sin 2k_1 - 2\mathrm{e}^{\gamma}\sin k_1}{\cos 2k_1 - 2\mathrm{e}^{\gamma}\cos k_1 + \mathrm{e}^{2\gamma}}. \tag{62}$$

These phase shifts provide the possibility of an alternative definition of the τ-function of the positon (48) as mentioned above. Computing the result of taking the limit $\gamma \to 0$ in (60)–(62) one obtains the τ-function

$$\tau_n^{\mathrm{pos}} = (2 - 2\cos k_1)\{\sin k_1[-(g_n + \tfrac{1}{2}) - \tfrac{1}{2}] + \tfrac{1}{2}\sin(2T_n + 2k_1)\}. \tag{63}$$

The essential difference between (63) and (49) is the factor $(2-2\cos k_1)$ which is obviously a consequence of the non-uniqueness of the solutions of the Toda lattice (1) as mentioned above. The additional terms appearing on the right-hand side of (63) can be eliminated by absorbing them into a new definition of the phases in T_n, g_n.

When comparing the results derived here with previous studies of positons for continuous integrable systems (cf [10]–[14]), it should be noted that the main technical difference between the continuous and semi-discrete case considered here is the fact that the Wronski determinant has been replaced by the Casorati determinant. Starting from this observation it is more or less straightforward to extend the present constructions to n-soliton–m-positon solutions and to higher-order positons derived in [10]–[14] for continuous evolution equations to the present case.

To conclude the technical remarks to the results presented here we note that existing studies of the finite difference Schrödinger equation as given, for example, in [17] could not be applied to the present case since they are based on potentials decaying faster than $1/n$. The general framework of (direct) scattering theory incorporating as a special case the slow decay of initial data as considered here can easily be worked out by extending the arguments used in [18]. The ansatz employed here, however, is completely independent of such a general theory since it is based on a purely algebraic approach.

The new concept of positons introduced above arose from several sources which should be mentioned here. We first recall that such a study contributes to fascinating attempts to extend possible applications of the inverse scattering method to the case of initial data slowly decaying at infinity. The second reason is the unusual spectral picture connected with positons: for fixed time they represent von Neumann–Wigner potentials generating positive eigenvalues embedded in the continuous spectrum (cf [10]–[14] and the references cited therein). One more intriguing reason of interest concerns applications of the subvariety of even multi-positon potentials to quantum many-body problems. Taken as a pair interacion between one-dimensional particles of equal masses, they lead to the absence of Fresnel waves in the large distance asymptotics of the n-particle wavefunction, thus extending the results obtained by Buslaev and Merkuriev for short-range reflectionless pair interactions [19].

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